Stability analysis of cosmological models through Liapunov's method

Tiago C. Charters [1], Ana Nunes [2], José P. Mimoso [3]

[1]*ISEL, R. Conselheiro Emídio Navarro,1, 1949-014, Lisboa, Portugal*
[2,3]*FCUL, Ed. C1, Campo Grande, 1749-016 Lisboa, Portugal*
[1,3]*CFNUL and* [2]*CMAF, Av. Prof. Gama Pinto, 2, 1649-003, Lisboa, Portugal*

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Abstract

We investigate the general asymptotic behaviour of Friedman-Robertson-Walker (FRW) models with an inflaton field, scalar-tensor FRW cosmological models and diagonal Bianchi-IX models by means of Liapunov's method. This method provides information not only about the asymptotic stability of a given equilibrium point but also about its basin of attraction. This cannot be obtained by the usual methods found in the literature, such as linear stability analysis or first order perturbation techniques. Moreover, Liapunov's method is also applicable to non-autonomous systems. We use this advantadge to investigate the mechanism of reheating for the inflaton field in FRW models.

1 Introduction

A major difficulty in analyzing general relativistic problems, namely in cosmology and black hole physics, stems from the fact that the relevant field equations are nonlinear. This seriously limits the possibility of obtaining exact solutions, and makes it difficult to assess the degree of generality of the behaviour and special features of those exact solutions which are known. On the other hand, there is an increasing realization of the importance of the asymptotic behaviour of models, which provides the relevant features to be compared with the physical data available for each era of the universe.

As a consequence, in recent years the emphasis in the study of cosmological models has drifted from the search for exact solutions to the analysis of the qualitative properties of the equations and of the long term behaviour of the solutions [1]. The main tools used in this context have been perturbation methods and qualitative theory, especially linear stability analysis. Liapunov's method has been used in relatively few instances [2, 3] as a means to prove stability.

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[1]email: charters@cii.fc.ul.pt

[2]email: anunes@lmc.fc.ul.pt

[3]email: jpmimoso@cii.fc.ul.pt

In this paper, we argue in favour of the use of Liapunov's method for the stability analysis of the possible attractors in a large class of cosmological models, as an alternative to linear approximation. One main advantage of the method is that it provides information not only about stability but also about the basins of attraction. Moreover, Liapunov's method is also applicable to non-autonomous systems [4, 5].

In its stronger version, Liapunov's method depends on the construction of a phase function F with the properties of being strictly decreasing along the orbits and having a minimum at the equilibrium point. Although stated in most introductory books in dynamical systems, this method is seldom used in the applications because it is difficult in general to find the form of the function F. However, there are some simple models in physics which indicate an obvious candidate for F such as in the case of the classical textbook application, the damped oscillator: the energy of the system is a positive definite form, and decreases along the orbits, thus verifying the condition of applicability of the method in its stronger version.

A wide class of cosmological models share with the latter example the same form and qualitative dynamics. The equations of motion for *all* these models can be generically written as

$$3\frac{\dot{a}^2}{a^2} = \sum_{i=0}^{N} \frac{1}{2}\dot{b}_i^2 + M(a, b_1, \ldots, b_N) \tag{1}$$

$$\ddot{b}_i = -3\frac{\dot{a}}{a}\dot{b}_i - \frac{\partial M}{\partial b_i}(a, b_1, \ldots, b_N), \qquad i = 1, 2, \ldots, N \tag{2}$$

where a is non-negative.

The degrees of freedom can be divided into two sets. The first corresponds to the global scale factor a which has, in the simpler models, a monotone behaviour in time, and the second set is formed by the quantities b_i, $i = 1, 2 \ldots, N$, which in many cases behave as damped oscillators and are thus especially well suited to the study and analysis through the construction of Liapunov functions.

The paper is structured as follows. In Section 2, we briefly recall the basic results underlying Liapunov's method. In Section 3 we shall consider various examples for which a Liapunov function may be constructed that coincides essentially with the energy of the damped oscillator modes. For each case, the relevant dynamical information provided by the method is presented. First, in Section 3.1, we study the basin of attraction of the de Sitter behaviour in Friedman-Robertson-Walker models with an inflaton field having a non-vanishing vacuum. We also assess the conditions for the existence of parametric resonance when the scalar field equation exhibits a periodic forcing term. In Section 3.2 we consider the specific model of single-field reheating [6, 7, 8, 9, 10] and show that the results derived on the parametric resonance define the admissible class of potentials. Next, in Section 3.3, we consider the convergence of scalar-tensor

gravity theories to general relativity in the case of Friedman-Robertson-Walker models [11, 12, 13]. Subsequently, in Section 3.4, we restrict ourselves to the simple case of diagonal Bianchi type-IX cosmologies with no matter and a positive cosmological constant and study the validity of the cosmic no-hair theorem for this model. We shall recover in a simple manner, for this model, the result proved in [14]. Finally, the new information provided in each case by Liapunov's method is summed up in Section 4.

2 Reminder of the basic theorems

Let us start by stating some general definitions.

Given a smooth dynamical system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, and an equilibrium point x_0, we say that a continuous function $F : \mathbb{R}^n \to \mathbb{R}$, in a neighborhood U of x_0, is a Liapunov function for the point x_0 if

(i) F is differentiable in $U \backslash \{x_0\}$,

(ii) $F(x) > F(x_0)$ and

(iii) $\dot{F}(x) \leq 0$ for every $x \in U \backslash \{x_0\}$.

The existence of a Liapunov function F guarantees the stability of x_0; furthermore, if the strict inequality $\dot{F}(x) < 0$ holds in $U \backslash \{x_0\}$ then x_0 is asymptotically stable (see for instance [5])

The neighborhood U is essentially constrained by condition (iii). So, when (iii) holds strictly, the method provides information not only about the asymptotically stability of the equilibrium point but also about its basin of attraction, which must contain the set U. This kind of information cannot be obtained neither by linear stability analysis nor by first order perturbation theory.

Again in the case when (iii) holds strictly, the existence of a Liapunov function for the equilibrium point x_0 has important consequences for the behaviour of time dependent perturbations of the above autonomous system. Consider the perturbed system

$$\dot{x} = f(x) + g(x, t), \tag{3}$$

where $g(x, t)$ is smooth and bounded. Malkin's theorem [4] states that the solutions $x(t; x(t_0), t_0)$ of (3) will remain for all time inside a prescribed neighborhood B_ϵ of x_0, provided that we pick initial conditions $x(t_0)$ sufficiently close to x_0 and that the magnitude of the perturbation $||g(x, t)||$ is sufficiently small for all $t > t_0$ and all $x \in B_\epsilon$. This theorem guarantees the stability of x_0 for any sufficiently small time dependent perturbation, but it does not imply that the solution $x(t; x(t_0), t_0)$ tends to x_0 when $t \to \infty$, since it does not even require that $g(x, t)$ should vanish as $t \to \infty$. If this is the case, then a related result also

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due to Malkin (see [4]) shows that an asymptotic property persists, in the sense that, as time increases, the neighborhood B_ϵ may be taken progressively smaller.

Liapunov's method can also be used in the more general setting of time dependent systems which are not perturbations of autonomous systems (see [4]). Given a smooth non-autonomous system $\dot{x} = f(x,t)$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and an equilibrium point x_0 such that $f(x_0,t) = 0$, if there exists a smooth function $F : U \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, U a neighborhood of x_0, such that $F(x_0,t) = F_0 = const$, $F(\cdot,t) - F_0$ is positive definite on U for every $t \in \mathbb{R}$ and $\dot{F}(x,t) \leq 0$ on $U \times \mathbb{R}$ then x_0 is stable.

This result is analogous to the weak version of the theorem for the autonomous case. In the non-autonomous case, however, the conditions for asymptotic stability are more restrictive. The point x_0 is asymptotically stable if, moreover, $F(x,t)$ is bounded from above, for all t, by a continuous increasing function of the distance $||x - x_0||$, and $\dot{F}(x,t)$ is negative definite in U for all $t \in \mathbb{R}$.

3   Stability analysis

3.1   The basin of attraction of de Sitter inflation

During inflation the leading contribution to the energy momentum tensor is given by the inflaton scalar field ϕ. The evolution of the Friedman-Robertson-Walker universes is described by

$$3\frac{\dot{a}^2}{a^2} + 3\frac{k}{a^2} = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{4}$$

$$\ddot{\phi} = -3\frac{\dot{a}}{a}\dot{\phi} - \frac{\mathrm{d}V}{\mathrm{d}\phi}(\phi) \tag{5}$$

where a is the (positive) scale factor of the universe, and $k \in \{-1,0,1\}$ distinguishes the various spatial curvature cases (notice also that we adopt units in which $8\pi G = 1 = c$ throughout the paper). Under some assumptions on the scalar field potential V and on the initial conditions ϕ_0, $\dot{\phi}_0$ and a_0, there exist solutions where the friction term $3H\dot{\phi}$ dominates in (4) over $\ddot{\phi}$ and the potential term in (5) dominates over the kinetic term [15, 16]. This is the inflationary stage, where the universe evolves with accelerated expansion. For definiteness, we shall consider the simplest model with $V(\phi) = \lambda + m^2\phi^2/2$, $\lambda \geq 0$.

With the change of variable $H = \dot{a}/a$ and $b = 1/a$ equations (4) and (5) become

$$3H^2 = -3kb^2 + \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{6}$$

$$\dot{b} = -Hb \tag{7}$$

$$\ddot{\phi} = -3H\dot{\phi} - \frac{\mathrm{d}V}{\mathrm{d}\phi}(\phi). \tag{8}$$

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For future use we shall also need

$$\dot{H} = kb^2 - \frac{1}{2}\dot{\phi}^2. \tag{9}$$

Our first goal in this section will be to apply Liapunov's method for autonomous systems to the characterization of the late time behaviour of the model associated to equations (6), (7) and (8). This system has an equilibrium point at $(\phi = 0, \dot{\phi} = 0, b = 0)$ corresponding to $a = \infty$ where $H = \sqrt{\lambda/3}$. Since $H = \dot{a}/a$, this equilibrium point corresponds to an infinite universe with exponential asymptotic expansion rate whenever $\lambda \neq 0$. We shall therefore call the point $(\phi = 0, \dot{\phi} = 0, b = 0)$ the de Sitter spacetime.

Let us prove that for $k = -1, 0$ any initially expanding model will approach the de Sitter spacetime asymptotically for any ϕ_0, $\dot{\phi}_0$ and b_0, and that for $k = 1$ the basin of attraction of the de Sitter attractor is given by $b_0 < \sqrt{\lambda/3}$.

From (6), for an initially expanding model, H can be written as a function of b, ϕ and $\dot{\phi}$ as

$$H(\phi, \dot{\phi}, b) = \frac{1}{\sqrt{3}}\left(-3kb^2 + \frac{1}{2}\dot{\phi}^2 + \lambda + \frac{1}{2}m^2\phi^2\right)^{1/2}. \tag{10}$$

Notice that the argument of the square root is always positive for k non-positive. For the case $k = 1$ this is also true if the initial value of the scale factor satisfies $b_0 < \sqrt{\lambda/3}$, because then b is a monotonically decreasing function of time.

The asymptotic stability of the de Sitter attractor is obtained, as in the case of the classic damped oscillator, by using a Liapunov function given by

$$F(b, \phi, \dot{\phi}) = \frac{3}{2}b^2 + \frac{1}{2}\dot{\phi}^2 + V(\phi) > 0. \tag{11}$$

From equations (7), (8) and (9) we have

$$\dot{F}(b, \phi, \dot{\phi}) = -3H\left(\dot{\phi}^2 + b^2\right) < 0, \tag{12}$$

since $H(\phi, \dot{\phi}, b) > 0$. Then, F given by (11) satisfies (i), (ii) and (iii) for equations (7), (8), and (iii) holds strictly. This means that for $k = -1, 0$ and $\lambda \geq 0$ the de Sitter spacetime is a global attractor, while for $k = 1$ and $\lambda > 0$ its basin of attraction contains all initial conditions $(\phi_0, \dot{\phi}_0, b_0)$ such that $b_0 < \sqrt{\lambda/3}$.

Consider now the following modification of the equations of motion

$$3H^2 = -3kb^2 + \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{13}$$

$$\dot{b} = -Hb \tag{14}$$

$$\ddot{\phi} = -3H\dot{\phi} - m^2\left(1 + \varepsilon f(t)\right)\phi, \tag{15}$$

where f is a bounded, continuous, periodic function with period close to the free oscillating period of the unperturbed equation (15).

We shall take equations (13), (14), (15), as a toy model for the study of the problem of reheating, or parametric resonance, and defer to the following subsection its application to a specific model.

Because (iii) holds strictly Malkin's theorem can be applied to this case. It states that the solutions of (13), (14) and (15), will remain for all time inside a prescribed neighborhood of the infinity attractor provided that we pick initial conditions sufficiently close to it and that the magnitude of the perturbation is sufficiently small for all t. This theorem guarantees the stability of the infinity attractor for any sufficiently small time-dependent perturbation, but it does not imply that the solution tends to it as time goes to infinity. Nevertheless, we can look for some hypothesis concerning H which will entail the asymptotic stability of the equilibrium point $\phi = \dot{\phi} = b = 0$. Let

$$F(b, \phi, \dot{\phi}) = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\left(3\alpha H(\phi, \dot{\phi}, b) + m^2\right)\phi^2 + \alpha\phi\dot{\phi} + \frac{3}{2}b^2, \qquad (16)$$

which is positive definite for $0 < \alpha < m$. We obtain, using (13), (14), (15),

$$\dot{F} = -\left(3H - \alpha\right)\dot{\phi}^2 \quad - \quad \alpha\left(m^2 - \frac{3}{2}\dot{H} + m^2\varepsilon f(t)\right)\phi^2 - m^2\varepsilon f(t)\phi\dot{\phi}$$
$$- \quad 3Hb^2. \qquad (17)$$

$\dot{F}$ will be negative definite if

$$(3H - \alpha)\left(m^2 - \frac{3}{2}\dot{H} + m^2\varepsilon f(t)\right) - \frac{1}{4}m^4\varepsilon^2 f^2(t) > 0. \qquad (18)$$

Since $\dot{H}(\phi = 0, \dot{\phi} = 0, b = 0) = 0$, this condition will hold for t large enough and ϵ small enough whenever $\lim_{t \to +\infty} H(t) > \alpha/3$.

There are two basic conclusions that can be obtained from these facts. First, parametric resonance can only occur when (18) does not hold, that is, parametric resonance can only occur if H tends to zero as time goes to infinity, because, otherwise, the Liapunov function (16), ensures that the equilibrium point $\phi = \dot{\phi} = b_0 = 0$ is asymptotically stable. This implies, for this toy model, that reheating can only be obtained from a potential that has a null minimum $V(0) = \lambda = 0$. Second, even if H tends to zero as time goes to infinity, parametric resonance can only be observed if ε is greater than some critical value because otherwise Malkin's theorem implies the stability of the infinity attractor.

3.2 Single field reheating

We now examine a specific model of reheating after inflation. According to the inflationary theory, almost all elementary particles populating the universe were

created during the process of reheating after inflation [7]. Reheating occurs due to particle production by the oscillating scalar field ϕ. In the simplest models, this field ϕ is the same inflaton field that drives inflation at the early stages of the evolution of the universe. After inflation, the inflaton oscillates near the minimum of its potential and this induces the quantum process of creation of particles [7]. At the root of this process is a phenomenon of parametric resonance which gives rise to the excitation of certain modes of the quantum fluctuations of fields [9, 17].

We shall consider a simple model with the scalar field potential of the form $V(\phi) = V_0\phi^n$, with $V_0 > 0$, characteristic of the chaotic inflation scenario, and of the original studies of reheating. Our purpose is to ilustrate the possibilities underlying the aplication of the results of the previous subsection to this model. We aim at determining under what conditions does the mechanism of resonance occur, avoiding the drastic assumption often made in the analytic treatment found in the literature which consists in neglecting the Hubble rate of expansion [6, 7, 9, 10]. Furthermore, we also wish to show that the theorem on parametric resonance requiring the damping term to be vanishing in time and the amplitude of the periodic perturbation in the scalar field equation to be bounded from below permits to constrain the form of the potential in the case under consideration.

We consider a FRW metric with non-positive curvature and shall focus on perturbations of the single scalar field ignoring the accompanying metric fluctuations. This is well justified on scales smaller than the Hubble distance, that is, satisfying $p \geq a\,H$, where p is the wave number of the perturbation, since relativistic effects play a lesser role on these scales [18]. On scales larger than the Hubble radius this approximation is more restrictive and, in particular, it has been recently shown to miss the resonant behaviour of the metric perturbations [19, 20, 21]. With this caveat we shall thus consider the set of equations of motion

$$3H^2 = -3kb^2 + \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{19}$$

$$\dot{b} = -Hb \tag{20}$$

$$\ddot{\phi} = -3H\dot{\phi} - nV_0\phi^{n-1} + b^2\nabla_{\vec{x}}^2\phi, \tag{21}$$

with $n > 1$ and $\vec{x}$ the spatial coordinates.

Following [10], we decompose the inflaton field into the sum of a classical inflaton field φ and a quantum fluctuation $\delta\phi$ in the form $\phi(\vec{x}, t) = \varphi(t) + \delta\phi(\vec{x}, t)$. We next concentrate in a specific mode of the fluctuation $\delta\phi$. This yields, keeping the first order terms in $\delta\phi_p$,

$$\ddot{\varphi} + 3H\dot{\varphi} + nV_0\varphi^{n-1} = 0 \tag{22}$$

$$\delta\ddot{\phi}_p + 3H\delta\dot{\phi}_p + \left[n(n-1)V_0\varphi^{n-2} + p^2b^2\right]\delta\phi_p = 0. \tag{23}$$

Notice that n should be strictly greater than two for the existence of resonant behaviour.

Let $d\tau/dt = b$ and denote the derivative $d/d\tau$ by a prime. The equations of motion become

$$3H^2 = -3kb^2 + \frac{1}{2}b^2\varphi'^2 + V(\varphi) \tag{24}$$

$$b' = -H \tag{25}$$

$$\varphi'' = 2\frac{b'}{b}\varphi' - nb^{-2}V_0\varphi^{n-1} \tag{26}$$

$$\delta\phi_p'' = 2\frac{b'}{b}\delta\phi_p' - \left[n(n-1)b^{-2}V_0\varphi^{n-2} + p^2\right]\delta\phi_p \tag{27}$$

with $n > 2$.

Let $T = \dot{\varphi}^2/2 = b^2\varphi'^2/2$ and $\gamma = 2T/(T+V)$. Since $2\overline{T} = n\overline{V}$ holds asymptotically [22], we have $\overline{\gamma} = 2n/(n+2)$ and a simple argument shows that $H \sim b^{3\overline{\gamma}/2}$ if $k = 0$, $H \sim b$ if $k = -1$. Consequently b'/b and $b^{-2}\varphi^{n-2}$ approach the infinity attractor as

$$b'/b \sim \begin{cases} b^{(2n-2)/(n+2)} & \text{if } k = 0 \\ b & \text{if } k = -1 \end{cases} \tag{28}$$

$$\varphi^{n-2}b^{-2} \sim b^{(4n-16)/(n+2)}. \tag{29}$$

From the previous discussion of our toy model we know that, in order to have parametric resonance, the damping term in (27) must tend to zero and the perturbing term in (27) must be bounded away from zero. The first condition is always satisfied, in view of (28) and of the fact that n must be greater than 2. From (29), the second condition is satisfied provided that $n \leq 4$. Notice that for $n = 4$ the term $\varphi^{n-2}b^{-2}$ in (27) oscillates with constant amplitude for all time. This is the case usually studied in the literature (see for example [7]). However, (29) shows that reheating should be more effective for potentials with exponents closer to 2 in the allowed interval $(2, 4]$.

3.3 General Relativity as a Cosmological attractor of Scalar-Tensor Gravity Theories

General scalar-tensor gravity theories are based on the Lagrangian [11, 12, 13]

$$L_\Phi = \Phi R - \frac{\omega(\Phi)}{\Phi}g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu} + 2U(\Phi) + 16\pi L_m \tag{30}$$

where R is the Ricci curvature scalar of a space time endowed with a metric $g_{\mu\nu}$, Φ is a scalar field, $\omega(\Phi)$ is a dimensionless coupling function, $U(\Phi)$ is a function of Φ, and L_m is the Lagrangian for the matter fields. They provide a most natural generalization of Einstein's general relativity, and their investigation enables a generic model-independent approach to the main features and cosmological implications of the unifications schemes. The archetypal of these theories

is Brans-Dicke theory [23, 24] for which $\omega(\Phi)$ is a constant and the most distinctive feature of the theories defined by (30) is the variation of the gravitational constant, since $G = \Phi^{-1}$.

Consider the homogeneous and isotropic Friedman-Robertson-Walker (FRW) universes and assume that the matter content of the universe is a perfect fluid with the usual equation of state $p = (\gamma - 1)\rho$, where $0 \leq \gamma \leq 2$. Here p and ρ are, respectively, the pressure and the energy-density measured by a comoving observer. Performing the conformal transformation of the metric [24, 25] $\tilde{g}_{ab} = (\Phi/\Phi_*)\, g_{ab}$, where Φ_* is a constant which normalizes the gravitational constant, to the so-called Einstein frame which corresponds to replacing the dynamical variables (a, t, Φ) with $\left(\sqrt{\frac{\Phi}{\Phi_*}}a, \sqrt{\frac{\Phi}{\Phi_*}}t, \varphi\right)$ where

$$\frac{d \ln \Phi}{d\varphi} = \sqrt{\frac{16\pi}{\Phi_*(2\omega(\varphi) + 3)}}\,, \tag{31}$$

the field equations reduce to the simple form

$$3\frac{\dot{a}^2}{a^2} + 3\frac{k}{a^2} = \frac{1}{2}\dot{\varphi}^2 + a^{-3\gamma}M(\varphi) + V(\varphi) \tag{32}$$

$$\ddot{\varphi} + 3\frac{\dot{a}}{a}\dot{\varphi} = -a^{-3\gamma}M'(\varphi) - \frac{dV}{d\varphi}(\varphi), \tag{33}$$

where $V(\varphi) = \Phi_*^2 U\left(\Phi(\varphi)\right) / \left(8\pi\Phi^2(\varphi)\right)$, $M(\varphi) = \mu\left(\Phi(\varphi)/\Phi_*\right)^{-2+3\gamma/2}$ and μ is a constant which fixes the initial conditions for the energy-density of the perfect fluid. We shall assume that $M(\varphi) \geq 0$ (which corresponds to having $G \geq 0$) and that $V(\varphi) = \lambda + \frac{1}{2}m^2\varphi^2$, i.e., that $V(\varphi)$ has a non-degenerate positive global minimum at the origin.

Let $H = \dot{a}/a$ and $b = 1/a$. It is easy to see that the former system is equivalent to

$$3H^2 = -3kb^2 + \frac{1}{2}\dot{\varphi}^2 + b^{3\gamma}M(\varphi) + V(\varphi) \tag{34}$$

$$\dot{b} = -Hb \tag{35}$$

$$\ddot{\varphi} = -3H\dot{\varphi} - b^{3\gamma}M'(\varphi) - \frac{dV}{d\varphi}(\varphi). \tag{36}$$

These equations define a dynamical system of dimension 3. We are interested in studying the stability of the equilibrium point at $a = +\infty$ corresponding to $b = 0$, $H = \sqrt{\lambda/3}$ and $\varphi = \dot{\varphi} = 0$, which is the de Sitter solution [13].

Let us prove that for $k = -1, 0$ any initially expanding model will approach the de Sitter spacetime asymptotically for any φ_0, $\dot{\varphi}_0$ and b_0, and that for $k = 1$ the basin of attraction of the de Sitter attractor is given by $b_0 < \sqrt{\lambda/3}$. Consider the initial conditions $H_0 > 0$ associated with an expanding model, and b_0, φ_0 and $\dot{\varphi}_0$

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arbitrary. Using the hypothesis $M(\varphi) \geq 0$, the expanding branch $H = H(\varphi, \dot{\varphi}, b)$ of (35)is given by

$$H = \frac{1}{\sqrt{3}} \left\{ -3kb^2 + \frac{1}{2}\dot{\varphi}^2 + b^{3\gamma} M(\varphi) + V(\varphi) \right\}^{1/2} > 0 . \qquad (37)$$

As for the case $k = 1$, we take again $H_0 > 0$ and $b_0 \leq \sqrt{\lambda/3}$ and, since, from (35), b is a strictly decreasing function of time, it is possible for these initial conditions to write $H = H(\varphi, \dot{\varphi}, b)$ as in (37).

The function defined by

$$F(b, \varphi, \dot{\varphi}) = \frac{1}{2}\dot{\varphi}^2 + \lambda + \frac{1}{2}m^2\varphi^2 + b^{3\gamma} M(\varphi) \qquad (38)$$

is a Liapunov function for the equilibrium point $(b, \varphi, \dot{\varphi}) = (0, 0, 0)$. Indeed using (34), (35), (36) and (37), it is easy to see that

$$\dot{F}(b, \varphi, \dot{\varphi}) = -3H \left(\dot{\varphi}^2 + \gamma b^{3\gamma} M(\varphi) \right) < 0 , \qquad (39)$$

and it is also straightforward to check that (i) and (ii) hold for F. Thus this proves that the equilibrium point $(b, \varphi, \dot{\varphi}) = (0, 0, 0)$ is asymptotically stable and shows that an attractor mechanism indeed exists when the matter content of the universe is a perfect fluid and the potential has a non-degenerate non-negative minimum. This means that for $k = -1, 0$ and $\lambda \geq 0$ the de Sitter spacetime is a global attractor, while for $k = 1$ and $\lambda > 0$ its basin of attraction contains all initial conditions $(\varphi_0, \dot{\varphi}_0, b_0)$ such that $b_0 < \sqrt{\lambda/3}$. A similar attracting mechanism should also exist in the non-isotropic case.

In general, linear stability analysis only provides information about the stability and asymptotic behaviour in a undetermined neighborhood of the equilibrium point. This can be easily seen in the following example. Consider the equations of motion (35), (36) for $\gamma = 4/3$ (for $\gamma \neq 4/3$ the linear stability method can not be applied without additional hypothesis on $M(\varphi)$). The eigenvalues associated with the de Sitter attractor are $\lambda_b = -\sqrt{\lambda/3}$, $\lambda_\varphi = -\sqrt{3\lambda} - \sqrt{3\lambda - 4m^2}$ and $\lambda_{\dot{\varphi}} = -\sqrt{3\lambda} + \sqrt{3\lambda - 4m^2}$. The linear approximation confirms the result obtained by the Liapunov method, the real part of the eigenvalues is always negative, and shows that there exist two basic behaviours for the field associated to the monotone approach of the scale factor to the de Sitter attractor: an oscillatory mode (damped oscillations) if $0 < \lambda < 4m^2/3$ and a monotone behaviour (over-damped) for the field if $\lambda \geq 4m^2/3$ which corresponds, respectively, to a stable focus and a stable node in the plane $(\varphi, \dot{\varphi}, b = 0)$. No information is obtained concerning the size of the basin of attraction for the de Sitter attractor with this method.

Also notice that when $\gamma < 1/3$ the second member of equation (36) is not differentiable at $b = 0$, which precludes the application of linear stability analysis in its simplest form. This fact emphasizes the advantage of the use of the Liapunov's method.

3.4 Asymptotic behaviour for diagonal Bianchi type-IX cosmologies in the presence of a positive cosmological constant without matter.

By definition, the general Bianchi type-IX spacetime has topology $\mathbb{R} \times \mathbf{S}^3$, with a simply transitive action of the isometry group $SU(2)$ on $\mathbf{S}^3$ spatial slices. The metric of a general Bianchi type-IX model can be put in the form [26]

$$ds^2 = -dt^2 + a^2(t) \sum_{i,j=1}^{3} \left[e^\beta\right]_{ij} d\omega^i d\omega^j , \qquad (40)$$

where ω^i, $i = 1, 2, 3$, are isometry invariant one-forms on the three-sphere, a is a scalar, and β is a traceless 3×3 matrix. Both a and β are functions of the proper time t only.

For a diagonal spacetime, let β_i, $i = 1, 2, 3$, denote the diagonal elements of the matrix β. Only two of these quantities are independent since $\beta_1 + \beta_2 + \beta_3 = 0$ on account of the tracelessness of β. We choose the independent variables to be [27]

$$b_1 = -\frac{1}{2\sqrt{6}}\beta_3 \qquad (41)$$

$$b_2 = \frac{1}{6\sqrt{2}}(\beta_1 - \beta_2). \qquad (42)$$

The vacuum Einstein equations with the cosmological constant take the form [26, 27]

$$3\frac{\dot{a}^2}{a^2} = \frac{1}{2}\left(\dot{b}_1^2 + \dot{b}_2^2\right) + \frac{1}{a^2}V(b_1, b_2) + \Lambda \qquad (43)$$

$$\ddot{b}_i + 3\frac{\dot{a}}{a}\dot{b}_i + \frac{1}{a^2}\frac{\partial V}{\partial b_i} = 0, \qquad i = 1, 2 \qquad (44)$$

with

$$V(b_1, b_2) = -e^{-\sqrt{\frac{2}{3}}b_1}\cosh(\sqrt{2}b_2) + \frac{1}{4}e^{-4\sqrt{\frac{2}{3}}b_1} \qquad (45)$$
$$+ \frac{1}{2}e^{2\sqrt{\frac{2}{3}}b_1}\left[\cosh(2\sqrt{2}b_2) - 1\right]$$

Let $H = \dot{a}/a$ and $b = 1/a$. It is easy to see that the former system is equivalent to

$$3H^2 = \frac{1}{2}\left(\dot{b}_1^2 + \dot{b}_2^2\right) + \Lambda + b^2V(b_1, b_2) = 0 \qquad (46)$$

$$\dot{b} = -Hb \qquad (47)$$

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$$\ddot{b}_i + 3H\dot{b}_i + b^2 \frac{\partial V}{\partial b_i} = 0, \qquad i = 1, 2. \tag{48}$$

These equations define a dynamical system of dimension 5. This system has two finite equilibrium points $b = 0$, $H = \sqrt{\Lambda/3}$ and $b = 2\sqrt{\Lambda/3}$, $H = 0$ both with $b_1 = b_2 = \dot{b}_1 = \dot{b}_2 = 0$. We are interested in the stability of the equilibrium point $b = 0$ associated with the de Sitter attractor.

Notice that near the minimum $(b_1, b_2) = (0, 0)$ the potential (45) takes the form

$$\begin{aligned} V(b_1, b_2) &= -\frac{3}{4} + \left(b_1^2 + b_2^2\right) + O_3(b_1, b_2) \\ &= -\frac{3}{4} + v(b_1, b_2), \end{aligned} \tag{49}$$

where $v(0, 0) = 0$.

We take $H_0 > 0$ and $b_0 \leq 2\sqrt{\Lambda/3}$. Since, from (47), b is a strictly monotone decreasing function with time the expanding branch of $H = H(b, b_1, b_2, \dot{b}_1, \dot{b}_2)$ can be written for these initial conditions as

$$H = \frac{1}{\sqrt{3}} \left\{ \frac{1}{2} \left(\dot{b}_1^2 + \dot{b}_2^2\right) + b^2 v(b_1, b_2) + \Lambda - \frac{3}{4}b^2 \right\}^{1/2} > 0 . \tag{50}$$

The Liapunov function for the equilibrium point $(b, b_1, b_2, \dot{b}_1, \dot{b}_2) = (0, 0, 0, 0, 0)$ will now be

$$F(b, b_1, b_2, \dot{b}_1, \dot{b}_2) = \frac{1}{2} \left(\dot{b}_1^2 + \dot{b}_2^2\right) + b^2 v(b_1, b_2) + \frac{3}{2}b^2 \geq 0 . \tag{51}$$

Indeed using the equations of motion we have

$$\dot{F} = -3H \left[\dot{b}_1^2 + \dot{b}_2^2 + \frac{2}{3}b^2 v(b_1, b_2) + b^2 \right] < 0 \tag{52}$$

and thus it is a simple matter to check that all the conditions (i), (ii) and (iii) are satisfied. Therefore the equilibrium point $(b, b_1, b_2, \dot{b}_1, \dot{b}_2) = (0, 0, 0, 0, 0)$ is asymptotically stable and the basin of attraction for de Sitter spacetime contains all initial conditions $\left(b_0, b_1(0), b_2(0), \dot{b}_1(0), \dot{b}_2(0)\right)$ such that $b_0 < 2\sqrt{\Lambda/3}$. In this case the linear stability analysis reveals only that the eigenvalues of the de Sitter attractor are all negative.

4   Summary and conclusions

In this paper we have applied the Liapunov's method to the study of the general asymptotic behaviour of cosmological models, namely: FRW cosmologies with

inflaton field; scalar-tensor gravity theories in FRW; diagonal Bianchi-IX vacuum cosmologies in the presence of a positive cosmological constant. For these models we recover through this method the well known stability result for the de Sitter solution and we show that the method also provides information about its basin of attraction.

Furthermore, we have applied Liapunov's method to another problem of a completely different nature, namely the analysis of the conditions for the existence of the parametric resonance mechanism underlying the reheating process. We have concluded that in order to have parametric resonance, the damping term in the quantum perturbation equation must tend to zero and the forcing term must be bounded away from zero. This implies that for the basic model with potentials of the form $V(\varphi) = V_0 \varphi^n$, reheating may occur only when $n \in (2, 4]$. As far as we know, albeit the simplicity of the reheating model considered, this conclusion is new and fully illustrates the extension to which Liapunov's method may be utilized to draw rigorous results in cosmology.

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